PMM U.S.S.R.,Vol.44,pp.306-310 0021-8928/81/3 0306 $7.50/0

UDC 62-50

ON A MEANS OF OPTIMAL CONTROL BY THE EXTREMAL AIMING METHOD*

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A way is suggested for the approximate realization of the extremal aiming method that essentially lowers the required rapidity of the computing device used in the control loop. The results of simulating quasi-optimal control processes for the flight path of an aircraft at landing, by the way proposed, are presented.

The direct application of the extremal aiming method /1/ for solving many practical problems is made difficult by the stringent requirements on the rapid action of the computing device used in the control loop. The rapidity of response requirements can be eased by taking into account the peculiarities of the object being controlled when the optimal control is synthesized on switching surfaces /2/. However, the switching surfaces have to be constructed in an $(n+1)$-dimensional space of the object's positions, which presents extreme difficulties not so much because of the large amount of computational work as owing to the complexity of presenting the results. A simple decision rule can sometimes be obtained /3—5/ with a piece-wise-linear approximation of the switching surfaces, but in generally the n dimensions of the vector of the object's state essentially limit the application field of these methods. In the present paper a way is suggested for realizing the extremal aiming method for a wider class of conflict control problems, when it is required to ensure a prescribed object state in a k-dimensional subspace of lower dimension than the state space. In this case the problem is reduced to the construction of switching surfaces in a $(k+1)$-dimensional space, which permits an essential lowering of computer rapidity requirement and to simplity the procedure for choosing the optimal control.

1. Statement of the problem. Let an object's motion be described by the equation

$$x^{\cdot} = A(t)x + B(t)u + C(t)v,\ x(0) = x^{\circ},\ t \in [0, \theta],\quad u(t) \in P(t),\ v(t) \in Q(t) \tag{1.1}$$

where x is the n-vector of state, u is the control r-vector, v is the perturbation m-vector, A, B, C are coefficient matrices, P and Q are time-dependent convex compacta. The control performance is evaluated by a terminal criterion of form

$$I = \max_l l'x(\theta) / \rho(l) \tag{1.2}$$

Here $l = (l_k' : 0')'$ is an n-dimensional vector whose last $n-k$ components are identically zero, l_k is the unit k-vector, $\rho(l) \neq 0$ is the support function of the convex and compact terminal set $M(\theta)$ in a k-dimensional subspace of the object's state space $\{x\}$, and the prime denotes transposition. Criterion (1.2) has the sense of a normed error; the guaranteed condition of control success is

$$I_0 = \max_v I < 1 \tag{1.3}$$

It is required to construct a position control strategy $u^*(t, x)$ minimizing the guaranteed bound (1.3).

2. Extremal aiming. We present the basic aspects of the extremal aiming method /1/, needed for the subsequent exposition. The attainability domains $G^{(1)}(\theta)$ and $G^{(2)}(\theta)$ for the motions of object (1.1) under the actions of control u and perturbation v, respectively, are constructed at instant θ for the realized postion $\{t, x\}$. Since criterion (1.2) depends only on the first k coordinates of vector x, the domains $G^{(1)}$ and $G^{(2)}$ are constructed in the corresponding k-dimensional subspace of space $\{x\}$. To the boundaries of the attainability domains correspond maximum values of the polar distances of the support hyperplanes, which, according to /1/, are determined by the relations

$$r_1^*(l, t) = \max_u \left[-\int_t^\theta l'X(\theta, \tau) B(\tau) u(\tau) d\tau\right], \qquad r_2^*(l, t) = l'X(\theta, t)x(t) + \max_v \int_t^\theta l'X(\theta, \tau) C(\tau) v(\tau) d\tau \tag{2.1}$$

The vector $[l'X(\theta, \tau)]' = s(\tau)$ corresponds to the solution of the equation adjoint of the homogeneous Eq. (1.1), i.e.,

$$ds / d\tau = -A'(\tau)s,\ t \leqslant \tau \leqslant \theta \tag{2.2}$$

with boundary condition $s(\theta) = l$. The predicted value of the performance index (1.2) is

*Prikl.Matem.Mekhan.,44,No.3,434-440,1980 determined by the expression

$$I(t) = \max_l [r_2^*(l, t) - r_1^*(l, t)] / \rho(l) = \max_l \varphi(l) = \varphi(l^*) \tag{2.3}$$

The extremal actions $u^*(t)$ and $v^*(t)$ are selected from the maximum conditions

$$s'(l^*, t) B(t) u^*(t) = \max_u [-s'(l^*, t) B(t) u(t)] \tag{2.4}$$

$$s'(l^*, t) C(t) v^*(t) = \max_v s'(l^*, t) C(t) v(t) \tag{2.5}$$

3. Synthesis of the optimal control.

We introduce the linear transformation

$$y(\theta, t) = X_k(\theta, t) x(t) \tag{3.1}$$

where $X_k(\theta, t)$ is a $(k \times n)$-matrix composed from the first k rows of the fundamental matrix of solutions of (1.1). With due regard to equalities $l'X(\theta, t) = l_k'X_k(\theta, t)$ and to (3.1) the second relation in (2.1) can be rewritten as

$$r_2^*(l, t) = l_k'y(\theta, t) + \max_v \int_t^\theta l'X(\theta, \tau) C(\tau) v(\tau) d\tau \tag{3.2}$$

The direction of l^* for the instant t, used when selecting $u^*(t)$ in accord with (2.4), is determined only by the relative disposition of domains $G^{(1)}$ and $G^{(2)}$ which, according to (3.2), is characterized by vector $y(\theta, t)$. The sign of control $u_i^*(t, x)$ essentially depends uniquely on l^* and, consequently, on the predicted terminal state $y(\theta, t)$ of the object, i.e.,

$$u_i^*(t, x) = U_i(t) \operatorname{sign} \{\vartheta_i(t, y)\} \tag{3.3}$$

holds. The function $\vartheta_i(t, y) = 0$ specifies a switching surface for the components of control vector u_i^* in space $\{t, y\}$, while U_i is the i-th component of vector U satisfying the constraint in (1.1) and ensuring the maximum in (2.4). The nature of surfaces ϑ_i is determined by the equation and the contraints in (1.1) and by the performance index (1.2), which enables us to obtain beforehand the functions ϑ_i by the mathematical simulation method and to make numerical approximations of them. Then during the control we only need to compute the coordinates of vector $y(\theta, t)$ and to determine its position relative to the switching surfaces, using the coefficients of the approximating expressions for ϑ_{ia}. Since the complexity of approximating function $\vartheta_i(t, y)$ grows as this function's domain increases, it is advisable to limit the domain of possible initial states of the object

$$x(0) \in G_0 \tag{3.4}$$

The boundaries of domain G_0 are determined by operation requirements; in particular, this can be a domain in the object's state space, for which a successful solution of the control problem is guaranteed under any perturbations satisfying the constraint in (1.1). When (3.4) is fulfilled the possible positions of vector $y(\theta, t)$ defining the predicted terminal state of the object lie in a bounded set Y in space $\{t, y\}$; therefore, it is enough to determine the function $\vartheta(t, y)$ only for positions $\{t, y\} \in Y$. Since the approximation of function $\vartheta(t, y)$ is practically always implemented inexactly, a certain difference in the control processes obtained here from the processes in the strictly optimal system is inevitable. To stress this, the solutions obtained by the approximate method are subsequently called quasi-optimal.

Since the vector $y(\theta, t)$ determines the predicted terminal state of the object in its proper motion from position $\{t, x\}$, the value of y can be obtained by integrating the homogeneous Eq. (1.1) on the interval $[t, \theta]$ with initial condition for $x(t)$. The coordinates of vector y can also be obtained by using transformation (3.1); in this case it is necessary to make a numerical approximation of the row-vectors $X^{[i]}(\theta, t)$ of matrix $X_k(\theta, t)$ as functions of argument t, that describe the object's proper motion in terms of coordinates x_i, $i = 1, \ldots, k$. Then the integration of the homogeneous Eq. (1.1) on interval $[t, \theta]$ is replaced by the computation of y by formula (3.1) with the use of the coefficients of the approximating expressions $X_a^{[i]}(\theta, t)$.

4. Example.

1°. The lateral motion of the average transport aircraft during the final stage of landing approach can be described by the following equation /6/:

$$x^{\cdot} = Ax + b\gamma_3 + cW_z, \quad t \in [0, \theta], \quad \theta = 15\text{ c}, \qquad x = (z\ z^{\cdot}\ \psi\ \psi^{\cdot}\ \gamma\ x_6)' \tag{4.1}$$

$$A = \left\| \begin{matrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -0.0762 & -5.34 & 0 & 9.81 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & -0.0056 & -0.392 & -0.0889 & -0.0378 & -0.17 \\ 0 & 0 & 0 & 0 & -1.0 & 0 \\ 0 & -0.0129 & -0.9016 & -0.2015 & -0.0869 & -0.89 \end{matrix} \right\|$$

$$b = \operatorname{col}\{0, 0, 0, 0.0378, 1.0, 0.0869\}, \qquad c = \operatorname{col}\{0, 0.0762, 0, 0.0056, 0, 0.0129\}$$

Here z is the lateral deviation of the aircraft's center of mass from the runway axis, ψ and γ are the yaw and bank angles, respectively, x_6 is an auxiliary variable. The admissible magnitudes of the control (specified bank γ_3) and of the perturbation (cross-wind velocity W_z) are bounded in absolute value

$$|\gamma_3(t)| \leqslant \mu(t) = 0.2613 - 0.0116\,t, \text{rad}; \qquad |W_z(t)| \leqslant \nu = 10 \text{ m/s} \tag{4.2}$$

The performance index (1.2) corresponds to the following (λ is the polar angle in the plane $z0z^{\cdot}$):

$$I = \max_\lambda [z(\theta)\cos\lambda + z^{\cdot}(\theta)\sin\lambda] / \rho(\lambda) \tag{4.3}$$

The support function of the terminal set $M(\theta)$ is described by

$$\rho(\lambda) = \begin{cases} \rho_1(\lambda), & \rho_1(\lambda) > \rho_0 \\ \rho_0, & \rho_1(\lambda) \leqslant \rho_0 \end{cases} \tag{4.4}$$

$$\rho_1(\lambda) = R_0 |\sin(\lambda - 0.426\pi)|, \quad R_0 = 20.5, \quad \rho_0 = 1.7$$

The switching surface of control γ_3^* was constructed using computer simulation of the control processes by the extremal aiming method. The attainability domains $G^{(1)}$ and $G^{(2)}$ were constructed at discrete instants from interval $[0, \theta]$ as enveloping domains of the p support lines on plane $z0z^{\cdot}$. The values of function $\Delta r(\lambda)$, giving the difference between the polar distances of domains $G^{(2)}$ and $G^{(1)}$, were determined by solving a system of $2n+p$ differential equations. The solutions of the first $2n$ equations of form (2.2), $s^{(1)}$ and $s^{(2)}$, correspond to the first two rows of the fundamental matrix of solutions of Eq. (4.1), while the remaining p equations have the form

$$ds_{2n+j} / d\tau = |(s^{(1)}\cos\lambda_j + s^{(2)}\sin\lambda_j)'c|\,\nu - |(s^{(1)}\cos\lambda_j + s^{(2)}\sin\lambda_j)'b|\,\mu(\tau), \quad s_{2n+j}(0) = 0, \quad 0 \leqslant \tau \leqslant \theta - t \tag{4.5}$$

$$j = 1, \ldots, p$$

The direction of extremal aiming λ^* for the instant t was determined by the condition for maximizing function (2.3), which has the form

$$\varphi(\lambda_j) = [y^{(1)}\cos\lambda_j + y^{(2)}\sin\lambda_j + \Delta r(\lambda_j)] / \rho(\lambda_j), \quad j = 1, \ldots, p \tag{4.6}$$

where $y^{(1)}$ and $y^{(2)}$ are the coordinates of the object's terminal state on plane $z0z^{\cdot}$.

The section of surface ϑ by the plane $t = t_i$ corresponds to a solution of Eq. (4.7) and the boundary of domain $Y(t_i)$, of Eq. (4.8):

$$[s^{(1)}\cos\lambda^* + s^{(2)}\sin\lambda^*]'b = 0 \tag{4.7}$$

$$y^{(1)}\cos\lambda^* + y^{(2)}\sin\lambda^* + \Delta r(\lambda^*) = \rho(\lambda^*) \tag{4.8}$$

When constructing the attainability domains the equation system (4.5) was integrated by the Hemming predictor-corrector method with step 0.1 s The values of function Δr were computed in the upper halfplane of $z0z^{\cdot}$ (because of the symmetry of domains $G^{(1)}$ and $G^{(2)}$ relative to the origin) for 60 values of angle λ. This ensures the determination of λ^* to within $\pm 1{,}5°$ and requires the integration of a system of 72 equations. The construction of function ϑ and of the boundary of domain $Y(t)$ with time step $\Delta\tau = 5$ s (see Fig.1 where only the upper halfplane of $z0z^{\cdot}$ is shown because of the symmetry of $\vartheta(t)$ and $Y(t)$ took about 1 min of processor time on the computer M-4030. For the synthesis of the quasioptimal control $\gamma_{3a}^*(t)$

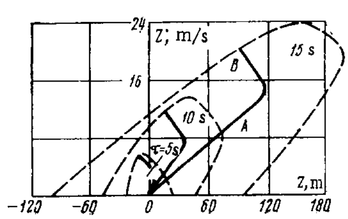

Fig.1

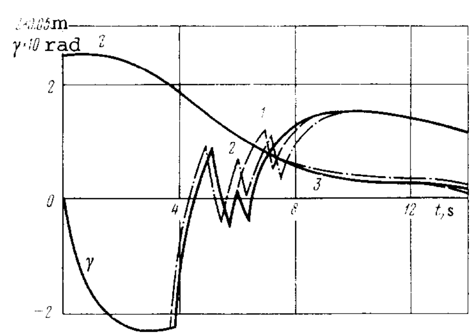

Fig.2

the following approximate description was adopted for the switching surface inside domain Y. At each instant $t \in [0, \theta]$ the section of surface ϑ was specified by two lines (A and B in Fig.1) whose equations are

$$y^{(1)} - K(t)\,y^{(2)} = 0, \qquad a_1 y^{(1)} + a_2 y^{(2)} - L(t) = 0, \quad a_1 = (1+\omega^2)^{-1/2}, \quad a_2 = -\omega(1+\omega^2)^{-1/2} \tag{4.9}$$

Here $K(t)$ and ω are the angular coordinates of lines A and B in plane $zOz^{\cdot}$; $L(t)$ is the polar distance of line B. The functions $K(t)$ and $L(t)$, as well as the variations of the vector-valued function $s^{(1)}(t)$ and $s^{(2)}(t)$, on the interval $[0, \theta]$ are well described by polynomials of a degree no higher than six (the errors of the uniform approximation are not more than 8%). To select the sign of w in control $\gamma_{3a}^*(t)$ it is sufficient to make two successive verifications of the signs of the left hand sides of (4.9) for values of $y^{(1)}$ and $y^{(2)}$ computed for instant t. Then the value of control $\gamma_{3a}^*(t)$ is determined as

$$\gamma_{3a}^*(t) = \mu(t)\,\text{sign}\,w \qquad (4.10)$$

To estimate the influence of the magnitude of the time step $\Delta_u t$ on the control performance the aircraft's motion was simulated under conditions of extremal perturbations with information discrimination of the control. The control $\gamma_{3a}^*(t)$ was chosen in accordance with (4.10) and a numerical procedure for constructing the attainability domains, presented above, was used for selecting the perturbation $W_z^*(t)$. The value λ^* was determined from the condition of maximizing the polar distances of the attainability domains with due regard to the variation of domain $G^{(1)}$ that corresponds to the object's motion on interval $[t, t+\Delta_u t]$ under the action of control $\gamma_{3a}^*(t)$ chosen at instant t. The extremal perturbation was chosen in accord with maximum condition (2.5) which in this case has the form

$$W_z^*(t) = \nu\,\text{sign}\,\{[s^{(1)}(\theta - t)\cos\lambda^* + s^{(2)}(\theta - t)\sin\lambda^*]'c\} \qquad (4.11)$$

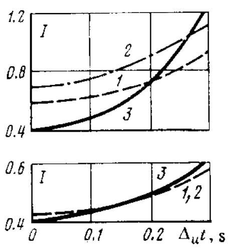

Fig.3

Simulation of the control processes (see Fig.2) was carried out with the choice of $\gamma_{3a}^*(t)$ by rule (4.10) for the cases when the object's predicted state is determined by integration of the homogeneous Eq. (4.1) on the interval $[t, \theta]$ (curves 1) and with the value of $y(\theta, t)$ computed by formula (3.1) (curves 2). The aircaft's motion was also simulated with the exact choice of $\gamma_3^*(t)$ by the extremal aiming method (curves 3). The dependence of the perfromance index (4.3) on the magnitude of $\Delta_u t$ for various initial states of the object (see Fig.3; its upper part corresponds to $z(0) = 50$ m, the lower to $z(0) = 0$) was obtained for the same cases (curves 1—3, respectively). For a small period $\Delta_u t$ the performance index of optimal control $\gamma_3^*(t)$ depends weakly on the initial state $x(0) \in G_0$ and does not exceed the index values obtained with the quasi-optimal control $\gamma_{3a}^*(t)$. An increase in $\Delta_u t$ causes a noticeable dependence of the estimate on $x(0)$, which is explained by the errors due to the discrete approximation of the extremal strategy $\gamma_3^*(t)$. This dependence is aggravated for the reason that the control game problem being analyzed corresponds to an essentially irregular case of a pursuit-evasion game when the size of domain $G^{(1)}$ diminishes with time significantly faster than that of $G^{(2)}$ and when function φ can have more than one maximum /1/. The instant of appearance of the second maximum of function φ and, consequently, its magnitude, depends on the initial state $x(0)$. In the case of a comparatively large magnitude $\Delta_u t$ the realisation of strategy $\gamma_3^*(t)$ differs somewhat from the optimal strategy when $\Delta_u t \to 0$; therefore, the estimate of the performance index of the quasi-optimal control $\gamma_{3a}^*(t)$ may not exceed the estimate of the index obtained for the optimal control $\gamma_3^*(t)$.

2°. To evaluate the effectiveness of the proposed control algorithm we determine the requirements on the rapidity and the memory of the controlling computer for the considered example of control of an aircraft's lateral motion at landing. We take it that the addition operation corresponds to an elementary machine operation, and that multiplication takes twice as long as addition, which is typical of the majority of modern computers /6/. When synthesizing the optimal control the greatest economy in time for the construction of the attainability domains is achieved by a simultaneous integration of the system of $2n + p$ Eqs.(4.5). With a step $\Delta\lambda = 3°$ $(p = 60)$ a system of 72 equations must be solved on each interval $\Delta_u t$. For an interval $\theta = 15$ s with a step of 0.1 s the general number of integration steps is 150. Let the numerical integration be carried out by a method requiring at each step a double computation of the right-hand sides of the system of equations (4.5). Their single computation requires $2n^2(1 + 2) + 2p(4n \cdot 2 + n + 1) = 6756$ elementary operations. Then the total number of operations is $2.03 \cdot 10^6$. With due regard to the additional operations on the choice of λ^* and on the formation of control $\gamma_3^*(t)$ and under the condition that $\Delta_u t = 0.1$ s we obtain a requisite rapidity of $2.04 \cdot 10^7$ operations per second. The volume of computer memory required is determined by the number of cells for storing the coefficients of Eq. (4.1) and of functions (4.2) and (4.4), as well as by the number of working cells for carrying out the integration, for choosing λ^*, etc., which adds up to about 150 memory cells.

When synthesizing the quasi-optimal control about 100 operations are required for computing the coefficients $K(t)$ and $L(t)$ in Eqs. (4.9) on each interval $\Delta_u t$. The determination of the values of $y^{(1)}$ and $y^{(2)}$ by integrating Eq. (4.1) with steps of 0.1 s requires the

$2n^2(1+2)\,150 = 3.24\cdot10^4$ operations. With $\Delta_u t = 0.1$ s the requisite computer rapidity for realizing this variant can be estimated to be $3.25\cdot10^5$ operations per second. In this case 70 cells are required for storing the coefficients of Eqs. (4.1) and (4.9) and 30 working cells are required for the integration and for choosing u^*. For the computation of the coordinates $y(\theta, t)$ by formula (3.1), $2n\cdot6(1+2) = 256$ elementary operations are required, which for $\Delta_u t = 0.1$s yields a requisite rapidity of 3160 operations per second. In this case 110 coefficients of the approximating polynomials of functions $s^{(1)}(t), s^{(2)}(t), K(t), L(t)$ must be stored and 20 working cells are required for choosing the control. Thus, the use of the proposed algorithm permits an essential lowering of the requirement on the controlling computer's rapidity (by more than $6\cdot10^3$ times for the example considered) without increasing the memory volume required.

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Translated by N.H.C.